लक्ष्यं पिद्यमानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 4th Semester Examination, 2023

MTMACOR08T-MATHEMATICS (CC8)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times 5 = 10$

(a) Let $f:[0,1]\to\mathbb{R}$ be defined by

$$f(0)=0,$$
$$f(x)=(-1)^n,\ \frac{1}{n+1}<x\le\frac{1}{n},\ n=1,2,3,\ldots$$

Show that f is integrable on [0, 1].

(b) Let $f:[0,2]\to\mathbb{R}$ be a function defined by

$$f(x)=2x,\quad 0\le x\le 1$$
$$=x^2,\quad 1<x\le 2$$

Show that f has no primitive although f is integrable on [0, 2].

(c) Find the values of p, if any, so that the integral

$$\int_1^\infty \frac{dx}{x^p} \text{ is convergent.}$$

(d) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty}\frac{(n+1)x^n}{(n+2)(n+3)}.$$

(e) Test the uniform convergence of the sequence of functions $\{f_n\}$ on [0, 1] defined by $f_n(x)=x^n(1-x),\ 0\le x\le 1$.

(f) Verify whether the series $\sum_{n=1}^{\infty}\frac{x}{n(n+1)}$ converges uniformly in $[0, a]$ where $a>0$.

(g) Justify true or false: The function $f(x)=\sin x,\ 0\le x\le\pi$, can be expressed as a Fourier cosine series.

(h) If the power series $\sum_{n=1}^{\infty}a_nx^n$ is convergent for all $x\in\mathbb{R}$ find the value of $\limsup_{n\to\infty}|a_n|^{\frac{1}{n}}$.

2. (a) (i) Prove that a monotone function f defined on a closed interval $[a, b]$ is integrable in the sense of Riemann. 2+2

(ii) Show that the function $f:[0, n] \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{[x]+1}, \quad 0 \le x \le n,$$

where $n \in \mathbb{N}$, $n > 1$, is R-integrable.

(b) If f be integrable on $[a, b]$ then show that the function F defined by 4

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b]$$

is continuous on $[a, b]$.

3. (a) Show that the integral 4

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx \text{ converges if and only if } m > 0,\ n > 0.$$

(b) Show that the integral $\int_0^\infty \frac{x^{p-1}}{1+x}dx$ is convergent only when $0 < p < 1$. 4

4. (a) If for each $n \in \mathbb{N}$, $f_n:[a, b] \to \mathbb{R}$ be a function such that $f_n'(x)$ exists for all $x \in [a, b]$; $\{f_n(c)\}_n$ converges for some $c \in [a, b]$ and the sequence $\{f_n'\}_n$ converges uniformly in $[a, b]$, then prove that the sequence $\{f_n\}_n$ converges uniformly on $[a, b]$. 4

(b) The function f_n on $[-1, 1]$ are defined by $f_n(x) = \frac{x}{1+n^2x^2}$. Show that $\{f_n\}$ converges uniformly and that its limit function f is differentiable but the equality $f'(x) = \lim_{n\to\infty} f_n'(x)$ does not hold for all $x \in [-1, 1]$. 4

5. (a) Let g be a continuous function defined on [0, 1]. For each n in $\mathbb{N}$ define $f_n(x) = x^n g(x)$, $x \in [0, 1]$. Find a condition on g for which the sequence $\{f_n\}$ converges uniformly. 4

(b) If the series $\sum f_n$ converges uniformly in an interval $[a, b]$ prove that the sequence $\{f_n\}$ converges uniformly to the constant function 0 in $[a, b]$. 4

6. (a) Prove that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$. 4

(b) Show that improper integral $\int_0^\infty \frac{\sin x}{x}dx$ is convergent. 4

7. (a) Let $\sum_{n=0}^\infty a_n x^n$ be a given power series and $\mu = \overline{\lim} |a_n|^{1/n}$. Then show that the series is everywhere convergent if $\mu = 0$. 4

(b) Assuming $\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$ for $-1<x<1$, obtain the power series expansion for $\tan^{-1}x$. Also deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$. 3+1

8. Show that the function $f:[-\pi,\pi]\to\mathbb{R}$ be defined by 2+3+1+2

$$f(x)=\begin{cases}\cos x & 0\le x\le\pi\\ -\cos x & -\pi\le x<0\end{cases}$$

satisfies Dirichlet's condition in $[-\pi,\pi]$. Obtain the Fourier co-efficients and the Fourier series for the function $f(x)$. Hence find the sum of the series

$$\frac{2}{1.3}-\frac{6}{5.7}+\frac{10}{9.11}-\cdots\cdots$$

9. (a) Let $f_n(x)=\frac{nx}{1+nx}$, $x\in[0,1]$, $n\in\mathbb{N}$. Then show that 4

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx=\int_0^1 \lim_{n\to\infty} f_n(x)\,dx,$$

but $\{f_n\}_n$ is not uniformly convergent on [0, 1].

(b) Prove that the even function $f(x)=|x|$ on $[-\pi,\pi]$ has cosine series in Fourier's form as 3+1

$$\frac{\pi}{2}-\frac{4}{\pi}\left\{\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\cdots\right\}$$

Show that the series converges to $|x|$ in $[-\pi,\pi]$.

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